Fixed Point Iterative Techniques – An Application to Fractals

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ABSTRACT

In recent days Fractals and study of their phenomenon have became an interesting and important concept in the field of Mathematics. ***Benoit Mandelbrot (1975),*** *known as father of fractal geometry was first, who originated the term* ***"Fractals"****. A fractal is a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced -size copy of the whole .These fractals can be typically created by using different iterative methods. This paper explores the various iterative techniques to generate Fractals like Julia sets, Mandelbrot sets, Koch curve etc.*

Keywords: *Fractals, Iterative Techniques, Julia Sets, Mandelbrot Sets, Koch Curve.*

1. INTRODUCTION

Benoit Mandelbrot proposed the concept of fractal in 1970s. Fractal has been applied in various areas such as science, economics, engineering, cultures and arts because of the universality of fractal phenomena. It provides a new and important analytical tool to explore the complexity of the real world. Most of the people things that the geometry of nature is based on simple figures such as lines, circles, conic sections, polygons, sphere, and quadratic surfaces and so on. For example, tires of the vehicle are circular in, Solar system moves around the sun in elliptical orbit. Poles are cylindrical, etc. But there are also so many examples of objects in nature which are complex and irregular that cannot be represented by classical geometry. Have we ever thought, what is the shape of a cloud? Can we describe the structure of plants and trees? Can we model the shape of mountain? All of these structures shows that the geometry does not depend on the simple figures like circle, straight line etc. The dynamic behavior in nature can also be complex and irregular.

To analyze many of these questions fractals and Mathematical chaos are appropriate tools. These fractals can be generated by Iterative Techniques. This paper gives the detail study of important fractals generated by one step iterative technique, two-step iterative technique, three-step iterative technique and four-step iterative technique.[1-2][3].

2. FRACTAL

A fractal is a never-ending process. Fractals are infinitely complex patterns that are self-similar across different scales. For self-similarity, a shape must be able to be divided into parts that are smaller copies which are more or less similar to the whole. Because of the smaller similar divisions of fractals, they appear similar at all magnifications. For example if we zoom on a picture of a mountain again and again we still see a mountain .This shows the self-similarity of fractals. However, while all

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fractals are self-similar, not all self-similar forms are fractals for example, the real line (a straight line) is formally self-similar but fails to have other fractal characteristics. Natural objects that approximate fractals to a degree include tree, seashells, clouds, mountain ranges, lightning bolts, coastlines, snowflakes, and various vegetables (cauliflower and broccoli).

3. ITERATIVE PROCESS

A process is called an iterative process or a successive approximation if we apply the formulae for again and again to get the desire results, using the result of previous steps in the next step. A mathematical fractal is based on an equation that undergoes iteration, a form of feedback based on recursion [4]. In 1890, Charles Emile Picard investigated an iterative process. Now Mathematician mentions this process as Picard iterations. This iteration process is well known to play a vital role in fixed point theory, fractals, numerical analysis, computational analysis and several other areas of applicable mathematics. In 1918, French mathematician Gaston Julia [5] investigated the iteration process of a complex function intensively and attained a Julia set, which is a landmark in the field of fractal geometry. Given a function $f(x)$, and a starting value x_0, one can generate a new value $x_1 = f(x_0)$. Using the same process, the next value x_2 is obtained by another step $x_2 = f(x_1) = f(f(x_0)$.This type of process is an iterative process that generates a sequence $x_0, x_1, \dots x_k$ where x_k represents the kth iterate obtained after applying the function f to x_0 k times. This sequence is known as an **orbit** of its starting point x_0.These iterative procedures are one way to obtain the self-similarity exhibited by fractals [3][6]

3.1 Picard Iterative process [7]: It is an example of One-step feedback Process.

Let X be a non-empty set and $f: X \to X$ such that $x_n = f(x_{n-1}), n = 1, 2, 3, \dots$
is characterized by Peano–Picard iterations. Also for a point $x_0 \in X$, the set of all iterates of point x_0
i.e. $O(f, x_0) = \{x_n : x_n = f(x_{n-1}), n = 1, 2, \dots\}$
called the Picard orbit (generally called the orbit of f).The Picard iteration is a very useful mathematical tool and has been developed particularly for the numerical solution of complex problems.

3.2 Mann iterative process [8]: It is an example of Two-step feedback Process. Let X be any subset of real or complex and $f: X \to X$ such that $x_{n+1} = (1 - c_n)\, x_n + c_n\, f(x_n), n > 0$
where $c_0 = 1, 0 < c_n < 1, and\ \sum_n c_n = \infty$ is known as Mann iterative process. Also for for a point $x_0 \in X$, a sequence in X of the form

$$x_1 = (1 - c_1)\, x_0 + c_1\, f(x_0),$$

$$x_2 = (1 - c_2)\, x_1 + c_2\, f(x_1),$$

$$\vdots$$

$$x_n = (1 - c_n)\, x_{n-1} + c_n\, f(x_{n-1})$$

$$\vdots$$

where $0 < c_n < 1$ and $\{c_n\}$ is convergent , called superior sequence of iterates.

3.3 Ishikawa iterative process [9]: It is an example of Three-step feedback Process. Let X be any subset of real or complex and $f: X \to X$ such that

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$$y_n = t_n f(x_n) + (1 - t_n)\, x_n$$

$$x_n = c_{n-1} f(y_{n-1}) + (1 - c_{n-1})\, x_{n-1}$$

where $0 \leq t_n, c_n \leq 1$ and sequences $\{t_n\}$ & $\{c_n\}$

are convergent to a non-zero numbers.
This process is known as Ishikawa iterative process.

3.4 Noor iterative process [10]: It is an example of four-step feedback Process. Let X be any subset of real or complex and $f: X \to X$ such that

$$z_n = (1 - c_n)\, x_n + c_n\, f(x_n);$$

$$y_n = (1 - b_n)\, x_n + b_n\, f(z_n)\,;$$

$$x_{n+1} = (1 - a_n)\, x_n + a_n\, f(y_n)\,;$$

$$n = 0, 1, 2, \ldots$$

where $0 \leq a_n, b_n, c_n \leq 1$ and the sequences $\{a_n\}, \{b_n\}$ & $\{c_n\}$ are convergent to a non-zero numbers .This process is known as Noor iterative procedure.
For initial point $x_0 \in X$, the set of all iterates of point x_0 such that

$$NO(f, x_0) = \{x_{n+1}:$$
$$x_{n+1} = (1 - a_n)\, x_n + a_n\, f(y_n)\,;$$

$$y_n = (1 - b_n)\, x_n + b_n\, f(z_n)\,;$$

$$z_n = (1 - c_n)\, x_n + c_n\, f(x_n)\,;$$

$$n = 0, 1, 2, \ldots\}$$

called the Noor orbit denoted by NO.

4. JULIA SETS

French mathematician Gaston Julia introduced the concept of iterative techniques and in 1919 he creates the Julia set. The Julia set defines the boundary between prisoner set and escape set [10]. The prisoner set is a collection of points inside the Mandelbrot set and escape set is a collection of points outside the orbit of Mandelbrot set. Julia sets, which are fractal in nature, can be obtained for quadratic, cubic, and higher degree polynomials, by using one-step iterative technique, two-step iterative technique, three-step iterative technique, four-step iterative technique .The following is the definition of Julia set for $F_c(z) = z^n + c, n = 1, 2, 3, \ldots$

4.1 Definition: The set of points K whose orbits are bounded under the function iteration of $F_c(z)$ is called the filled Julia set. Julia set of F_c is the boundary of the filled Julia set K. The boundary of a set is the collection of points for which every neighborhood contains an element of the set as well as an element, which is not in the set [11][12].

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The following theorem gives the general R-superior escape criterion for Julia sets and its corollaries further shows the escape criterion for computational purposes using Picard orbit.

Theorem 1. Suppose $|z| \geq |c| > 2$, where c is in the complex plane. Then we get $|F_c^n(z)| \to \alpha$ as $n \to \infty$

Corollary 1. Suppose $|c| > 2$. Then the orbit of 0 escapes to infinity under $F_c^n(z)$.

Corollary 2. Suppose $|z| > \max\{|c|, 2\}$.Then $|F_c^n(z)| > |(1+\lambda)^n|z|$ and so $|F_c^n(z)| \to \infty$ as $n \to \infty$, where λ is a positive number.

Corollary 3. Suppose for some $k \geq 0$, we have $|F_c^k(z)| > \max\{|c|, 2\}$.

Then $|F_c^{k+1}(z)| > (1+\lambda)|F_c^k(z)|$, so $|F_c^n(z)| \to \alpha$ as $n \to \infty$.

The Mandelbrot and Julia Set fractals, demand the use of complex numbers to create the basic Mandelbrot (or Julia) set. For this one uses the equation $f(Z) = Z^n + C$, where both Z and C are complex numbers. In 2004 M. Rani and V. Kumar [13], introduced the superior Julia and Mandelbrot set (two-step feedback Process) using Mann iteration Technique. In 2010, Chauhan [14], created new Julia set and Mandelbrot set in the study of fractal geometry. The following theorem gives the general R-superior escape criterion for Julia sets and its corollaries further shows the escape criterion for computational purposes using R-Superior orbit.

Theorem 2. **(**General escape criterion**)** for the function $F_c^n(z) = z^n + c, n = 1, 2, 3, \ldots$ where $0 < s \leq 1, 0 < t \leq 1$ and c is the complex number. Define

$$z_1 = (1-s)z + sF_c^n(z)$$

$$\vdots$$

$$z_n = (1-s)z_{n-1} + sF_c^n(z_{n-1})$$

Then, the general escape criterion is

$$\max\{|c|, (2/s)^{1/n-1}, (2/t)^{1/n-1}\}$$

Corollary 1. Suppose that $|c| > (2/s)^{1/n-1}$ and $|c| > (2/t)^{1/n-1}$ exits. Then R-Superior orbit *RSO* $(F_c^n, 0, s_n, t_n)$ escape to infinity.

Corollary 2. Suppose that $|z_k| > \max\{|c|, (2/s)^{1/k-1}, (2/t)^{1/k-1}\}$ for some $k \geq 0$.Then $|z_{k+1}| > \gamma|z_k|, so$

$|z_n| \to \infty \; as \; n \to \infty$. This gives the general escape criterion for computing R-superior Julia sets via the function of the form

$$F_c^n(z) = z^n + c, n = 1, 2, 3, \ldots$$

The following figures represent the R-superior Julia sets for quadratic and cubic function using three step iterative techniques.

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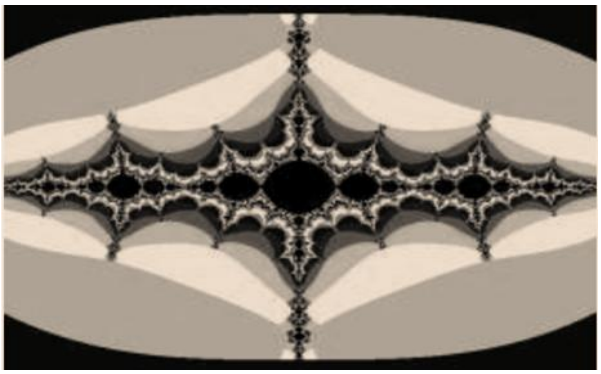

Figure 1: R-Superior Julia Sets for Quadratic

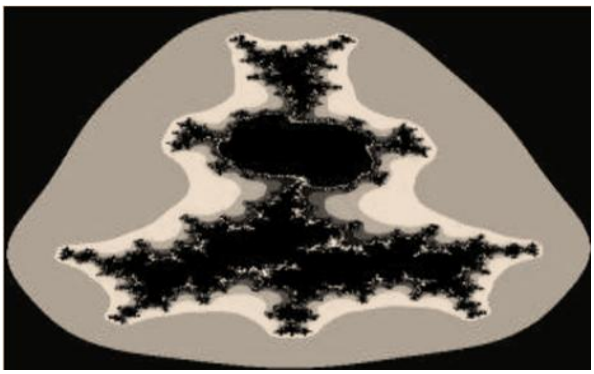

Figure 2: R-Superior Julia sets for Cubic

Recently, Ashish and R. Chugh (2014) [15] generated new Julia sets via Noor iterates (example of four-step feedback process). The following theorem gives the general escape criterion for Julia sets in Noor iterates and its corollaries further presents the escape criterion for computational purposes using Noor orbit.

Theorem 3. **(**General escape criterion**)** Assume the function $F_c(z) = z^n + c, n = 1, 2, 3, \ldots$ where$0 < r < 1, 0 < s < 1, 0 < t < 1$ and c is the complex number. Define

$$z_1 = (1-r)z + rF_c(z)$$

$$z_2 = (1-r)z_1 + rF_c(z_1)$$

$$z_n = (1-r)z_{n-1} + rF_c(z_{n-1})$$

Then, the general escape criterion is

max $\{|c|, (2/r)^{1/n-1}, (2/s)^{1/n-1}, (2/t)^{1/n-1}\}$

Corollary 1. Suppose that $|c| > (2/r)^{1/n-1}$, $|c| > (2/s)^{1/n-1}$ and $|c| > (2/t)^{1/n-1}$ exits. Then Noor orbit *NO* $(F_c, 0, r, s, t,)$ escape to infinity.

Corollary 2. (Escape criterion) Suppose that $|z_k| >$ max$\{|c|, (2/r)^{1/k-1}, (2/s)^{1/k-1}, (2/t)^{1/k-1}\}$ for some $k \geq 0$. Then $|z_k| > \gamma |z_{k-1}|, so$

$|z_n| \to \infty \; as \; n \to \infty$.

This corollary represents the algorithm to generate Julia set of the function $F_c(z) = z^n + c$.The following figures represent the Julia sets for quadratic and cubic function generated by four step iterative technique.

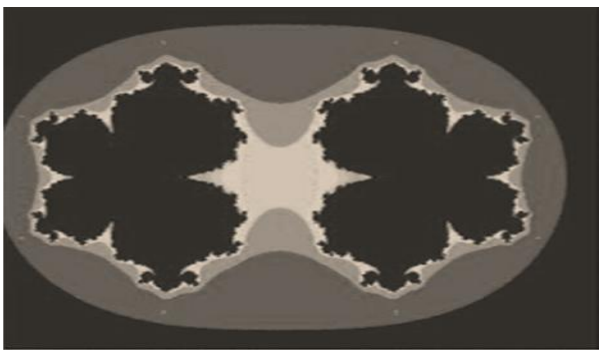

Figure 3: Quadratic Julia set using Noor Iterates

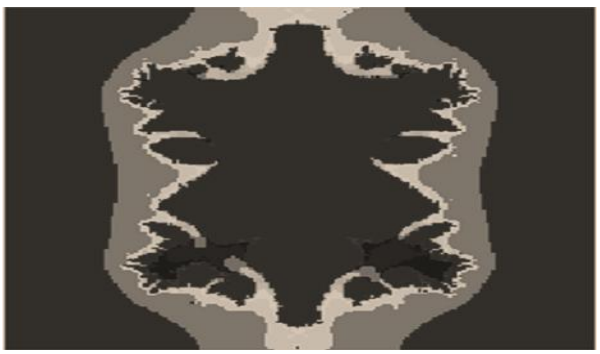

Figure 4: Cubic Julia set using Noor Iterates

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5. MANDELBROT SETS

In 1982, Mandelbrot [5] extended the work of Gaston Julia and introduced the Mandelbrot set, a set of all connected Julia sets. Similar to Julia Sets, Mandelbrot set, which are also fractal in nature, can be obtained for quadratic, cubic, and higher degree polynomials, by using one-step iterative technique, two-step iterative technique, three-step iterative technique, four-step iterative technique.

5.1 Definition: The Mandelbrot set M consists of all parameters c for which the filled Julia set of F_c is connected, i.e.

M= $\{c\epsilon C : K(F_c) \text{ is connected}\}$

In fact, M gives large amount of information about the structure of Julia sets. The R-Superior Mandelbrot set RSM for the Quadratic $F_c = z^2 + c$ is defined as the collection of all $c\epsilon C$ for which the orbit of the point 0 is bounded, i.e. RSM= $\{c\epsilon C : F_c^n(0);\ n = 0, 1, 2, \ldots$ is bounded$\}$. We select the initial point 0, as 0 is the only critical point of F_c [16]. The escape criterions studied above for Julia set are applicable in the generation of R-superior Mandelbrot sets and Mandelbrot sets Noor orbit.

Escape Criterion (R-Superior Mandelbrot set) The General escape criterion of R-superior Mandelbrot set for $F_c(z) = z^n + c$ is given by $|z_k| > \max\{|c|, (2/s)^{1/k-1}, (2/t)^{1/k-1}\}$ for some $k \geq 0$. The following figures represent the Mandelbrot sets for quadratic and cubic function generated by three step iterative techniques.

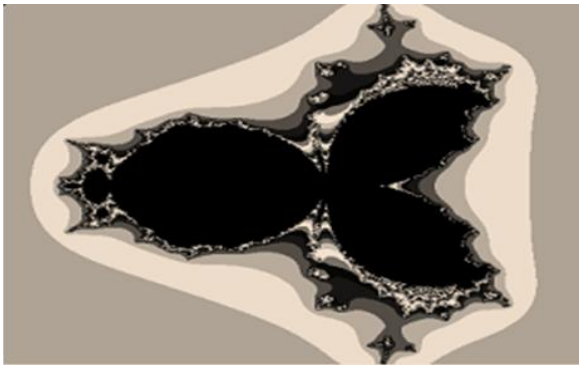

Figure 5: Quadratic R-Superior Mandelbrot Set

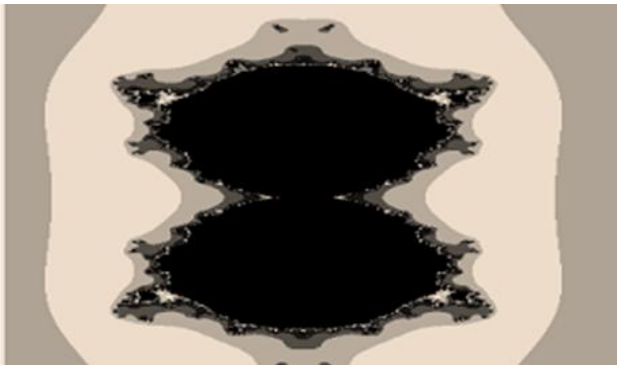

Figure 6: Cubic R-Superior Mandelbrot Set

Escape Criterion (Noor Mandelbrot set)

The General escape criterion of Noor Mandelbrot set for $F_c(z) = z^n + c$ is given by $|z_k| > \max\{|c|, (2/r)^{1/k-1}, (2/s)^{1/k-1}, (2/t)^{1/k-1}\}$ for some $k \geq 0$.The following figures represent the Mandelbrot sets for quadratic and cubic function generated by four-step iterative technique.

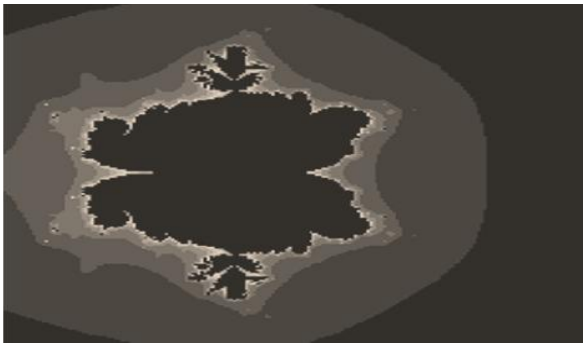

Figure 7: Quadratic Noor Mandelbrot Set

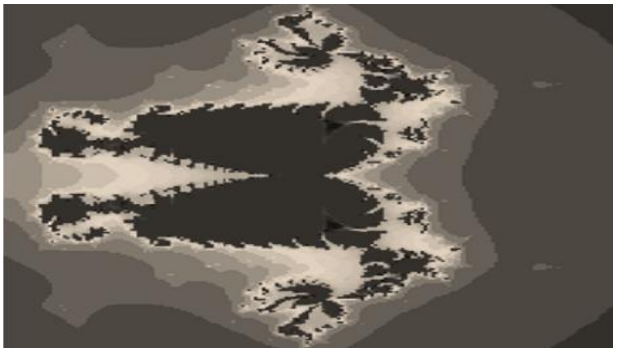

Figure 8: Cubic Noor Mandelbrot Set

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Mandelbrot set given by Mandelbrot and its relative object Julia set have become a wide and important area of research due to their beauty and complexity of their nature .The major difference in the Julia set and the Mandelbrot set is the iterative technique in which the function is iterated. The Mandelbrot set iterates $z = z^n + c$ with z always starting at 0 and varying the c value. The Julia set iterates $z = z^2 + c$ for a fixed c value and varying the values of z. In other words, the Mandelbrot set is in the parameter space, or the c-plane, while the Julia set is in the dynamical space, or the z-plane.

6. CONCLUSION

In this paper, an application of fixed point iterative techniques to generate fractals, using one-step iterative technique, two-step iterative technique, three-step iterative technique, four-step iterative technique, have been given. Different Julia sets and Mandelbrot sets generated above are examples of fractals. Through these iterative techniques more important fractals like Koch curve, Gasket etc. can also be generated.

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